

# What is the number of decompositions of torus into given number of regions by unions of geodesics?

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## Abstract

We prove some preliminary results concerning two questions of O.Karpenkov:

- (1) What is the number of decompositions (up to  $SL(2, \mathbb{Z})$ ) of two-dimensional torus into given number  $f$  of regions by unions of  $n$  geodesics?
- (2) On the plane there are  $n$  circles not in general position, every pair of circles has at least one common point. What is the set of all possible numbers of regions?

## Introduction

Let us consider a flat two-dimensional torus  $T$  (quotient space of real euclidean plane for an action of lattice — abelian group with two generators). In a fixed homology basis on torus  $T$  a closed oriented geodesic is defined (up to parallel shifting) by a pair of coprime integers. The matrix of changing from a homology basis to any other is an integer  $2 \times 2$  matrix with determinant  $\pm 1$ . We shall consider arrangements of nonoriented geodesics up to changing homology basis. Let  $f$  be the number of connected components of the complement in two-dimensional torus  $T$  to the union of  $n$  geodesics. The set  $F(n)$  of all possible numbers  $f$  for given  $n > 1$  is the following (see [1])

$$F(n) = \{n - 1, n\} \cup \{m \in \mathbb{N} \mid m \geq 2n - 4\}.$$

Let  $t_i$  be the number of intersection points, which are incident to  $i$  geodesics of the arrangement. If not all geodesics of the arrangement are parallel, then  $f = \sum_i (i - 1)t_i$ . For example, geodesics of types  $(1, 0)$ ,  $(0, 1)$ ,  $(k, 1)$  form  $k$  or  $k + 1$  regions if they intersect in one point or not. If in arrangement of geodesics  $\gamma_1, \dots, \gamma_n$  are at least two non-parallel, then

$$f \leq \sum_{i < j} |\gamma_i \cap \gamma_j|$$

where  $|\gamma_i \cap \gamma_j|$  is the number of intersection points of non-parallel geodesics. For arrangements of general position the inequality turns to equality.

## Question for torus

In connection with the theory of high-dimensional chain fractions O.Karpenkov asked:

What is the number of decompositions of two-dimensional torus into given number  $f$  of regions by unions of  $n$  geodesics?

**Lemma 1.** *If two geodesics intersect in  $k$  points in the two-dimensional flat torus, then we may change bases so that geodesics will be of type  $(1, 0)$  and  $(x, k)$ , where integer  $1 \leq x \leq k - 1$  is such that  $\gcd(x, k) = 1$  and is defined uniquely up to change  $x \leftrightarrow k - x$ .*

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**Lemma 2.** For  $f = n$  and  $f = n - 1$  there is a unique arrangement of  $n$  geodesics in the two-dimensional torus which divides torus into  $f$  regions. The number  $f = 2n - 4$  is realised as the number of regions by  $n - 1$  arrangements for  $n \geq 7$  ( and at least by 8 arrangements for  $n = 6$ ).

*Proof.* Let us take  $n - 2$  geodesics of type  $(1, 0)$  a geodesic of type  $(a, 1)$  and a geodesic of type  $(0, 1)$ , where  $0 \leq a \leq n - 2$ , and all intersection points of the last two geodesics are incident to some of the first  $n - 2$  geodesics.  $\square$

Let  $m$  be the maximal number of parallel (homologically equal) geodesics in an arrangements.

**Lemma 3.** If  $f \leq cn^{\frac{6}{5}}$ , then  $m \geq n - \frac{f}{n} + O(1)$ , for suitable positive coonstant  $c$ .

**Corollary 1.** For  $f \leq cn^{\frac{6}{5}}$  in arrangement of geodesics almost all geodesics are homologically equal and so the number of arrangements which realize  $f$  as the number of regions may be counted explicitly.

### Question for circles in the plane

O.Karpenkov asked the following “On the plane there are  $n$  circles not in general position, every pair of cicles has at least one common point. What is the set of all possible numbers of regions for given  $n$ ?”

For comlete solution of this problem one need to determine the possible number of tangent points of  $a$  circles and special arrangements of  $n - a$  lines for  $n > ca^2$ .

Let us denote by  $C_n$  the set of numbers  $f$  which are formed by  $n$  circles in the plane not in general position such that every two circles have at least one common point. Let us denote by  $L_n$  the set of numbers of regions in the plane, formed by  $n$  distinct lines not in general position (without any requirements on intersection points). Let  $m$  be the maximal number of circles, incident to one point.

**Lemma 4.** We have  $C_n \supseteq L_n$ .

*Proof.* Let us take any arrangement of lines in the plane with  $f \in L_n$ , make an inversion and get the suitable arrangement of  $n$  circles with  $f$  regions.  $\square$

If  $m = n$ , i.e. all circles have one common point, then the number of regions  $f \in L_n$ .

**Lemma 5.** If  $m = n - 1$ , then  $f$  may be any number of the sets  $L_{n-1} + 2n - 2$ ,  $L_{n-1} + 2n - 3$ ,  $L_{n-1} + 2n - 4$ , and this list of possibilities is uncomplete (here we sum a number to every element of  $L_{n-1}$ ).

The numbers  $3n - 4, 4n - 4 \in C_n$  and  $3n - 4, 4n - 4 \notin L_n$ , We have

$$L_n = \{n + 1, 2n, 3n - 3, 3n - 2, 4n - 8, 4n - 7, 4n - 6, 4n - 5, 5n - 15, \dots\}$$

**Conjecture 1.** The set  $C_n$  contains all integers between  $\frac{n(n-1)}{2} + 1$  and  $n(n - 1) + 2$ , which are the maximal elements of  $L_n$  and  $C_n$  correspondingly.

## References

- [1] I. N. Shnurnikov, On the number of regions formed by arrangements of closed geodesics on flat surfaces, *Math. Notes* **90**, N 3 – 4 (2011), 619 – 622.